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THE ABELIANIZATION OF THE CONGRUENCE IA-AUTOMORPHISM GROUP OF A FREE GROUP

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Dedicated to Professor Kojun Abe on the occation of his 60th birthday

ABSTRACT. In this paper we consider the abelianizations of some normal subgroups of the automorphism group of a finitely generated free group. Let F_n be a free group of rank n. For $d \geq 2$, we consider a group consisting the automorphisms of F_n which act trivially on the first homology group of F_n with $\mathbf{Z}/d\mathbf{Z}$-coefficients. We call it the congruence IA-automorphism group of level d and denote it by $IA_{n,d}$. Let $IO_{n,d}$ be the qutient group of the congruence IA-automorphism group of level d by the inner automorphism group of a free group. In this paper we determine the abelianization of $IA_{n,d}$ and $IO_{n,d}$ for $n \geq 2$ and $d \geq 2$. Furthermore, for $n = 2$ and odd prime p, we compute the integral homology groups of $IA_{2,p}$ for any dimension.

1. INTRODUCTION

Let F_n be a free group of rank n, and $\mathrm{Aut}\,F_n$ the automorphism group of the free group F_n. We denote the abelianization of F_n by H. The abelianization homomorphism $F_n \to H$ induces a surjective homomorphism $\rho : \mathrm{Aut}\,F_n \to GL(n, \mathbf{Z})$. In this paper we consider the abelianization of the preimages of the congruence subgroups of $GL(n, \mathbf{Z})$ by ρ. For $n \geq 2$ and $d \geq 2$, let $GL(n, d)$ be the general linear group over $\mathbf{Z}/d\mathbf{Z}$, and $\pi_d : GL(n, \mathbf{Z}) \to GL(n, d)$ the natural homomorphism induced by the mod reduction d. We call the kernel $\Gamma(n, d)$ of π_d the congruence subgroup of $GL(n, d)$ of level d. Classically, the congruence subgroups $\Gamma(n, d)$ have been studied by many authors, and there is a broad range remarkable results of them. In particular, it is well known that for $n \geq 3$ and odd prime integer p, Lee and Szczarba [10, Theorem 1.1] determined the structure of the abelianization of the congruence subgroup $\Gamma(n, p)$. More precisely, they showed that it is isomorphic to the Lie algebra $\mathfrak{sl}_n(\mathbf{F}_p)$ of trace-zero matrices over $\mathbf{F}_p$ as an $SL(n, \mathbf{F}_p)$-module where $\mathbf{F}_p$ is the finite field of order p.

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In this paper we are also interested in the kernel IA_n of the natural map ρ. We call it the IA-automorphism group of F_n. Then we have an exact sequence

$$1 \to IA_n \to \mathrm{Aut}\, F_n \xrightarrow{\rho} GL(n, \mathbf{Z}) \to 1.$$

This exact sequence plays important roles in the study of $\mathrm{Aut}\, F_n$. Although Magnus [11] obtained a finitely many generating set of IA_n for $n \geq 3$, (See Subsection 2.1.) it is not known whether IA_n is finitely presented or not for $n \geq 4$. We remark that Krstić and McCool [9] showed that IA_3 is not finitely presentable.

For a group G, we denote the abelianization of G by G^{ab}. Cohen-Pakianathan [2, 3], Farb [4] and Kawazumi [8] independently showed that the abelianization IA_n^{ab} of IA_n is a free abelian group of rank $n^2(n-1)/2$. More precisely, it is isomorphic to $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $GL(n, \mathbf{Z})$-module where H^* is the dual group $\mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ of H. We remark that the $GL(n, \mathbf{Z})$-module structure of IA_n^{ab} is determined by using the first Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Here we consider subgroups of $\mathrm{Aut}\, F_n$ which corresponds to the congruence subgroup of $GL(n, \mathbf{Z})$. Let $IA_{n,d}$ be the kernel of $\pi_d \circ \rho : \mathrm{Aut}\, F_n \to GL(n, d)$, and call it the congruence IA-automorphism group of a free group F_n of level d. The first aim of this paper is to determine the structure of the abelianization of $IA_{n,d}$. Then our first result is

Theorem 1.1. *For $n \geq 2$ and $d \geq 2$,*

$$IA_{n,d}^{\mathrm{ab}} \simeq (IA_n^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \bigoplus \Gamma(n, d)^{\mathrm{ab}}.$$

In Section 3, we prove this theorem using the "extended" Johnson homomorphism, introduced by Kawazumi [8], constructed from the $\mathbf{Z}/d\mathbf{Z}$-valued Magnus expansion of $\mathrm{Aut}\, F_n$. Considering the result of Lee and Szczarba [10] stated above, we see that for any odd prime p, the abelianization of IA_n is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n-1)(n^2+2n+2)}$ as an abelian group.

Next we consider the outer automorphism group of a free group and the images of IA_n and $IA_{n,d}$ by a natural projection. An automorphism ι of F_n is called an inner automorphism of F_n if there exists some element $y \in F_n$ such that $x^{\iota} = y^{-1}xy$ for any $x \in F_n$. Then the group $\mathrm{Inn}\, F_n$ of inner automorphisms of F_n is a normal subgroup of $\mathrm{Aut}\, F_n$. Let $\mathrm{Out}\, F_n$ be the quotient group $\mathrm{Aut}\, F_n/\mathrm{Inn}\, F_n$. The groups $\mathrm{Inn}\, F_n$ and $\mathrm{Out}\, F_n$ are called the inner automorphism group and the outer automorphism group of the free group respectively. We define a group IO_n to be the quotient group of IA_n by $\mathrm{Inn}\, F_n$. The group IO_n is the kernel of the natural map $\bar{\rho} : \mathrm{Out}\, F_n \to GL(n, \mathbf{Z})$ induced by ρ. It is also known that the abelianization IO_n^{ab} of IO_n is given by $IO_n^{\mathrm{ab}} \simeq (H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H$. (See [8, Theorem 6.2].) For $n \geq 2$ and $d \geq 2$, we define $IO_{n,d}$ to be the quotient group of $IA_{n,d}$ by $\mathrm{Inn}\, F_n$. The group $IO_{n,d}$ is the kernel of $\pi_d \circ \bar{\rho}$. The second aim of this paper is to determine the structure of the abelianization $IO_{n,d}^{\mathrm{ab}}$. The result is

Theorem 1.2. *For $n \geq 2$ and $d \geq 2$,*

$$IO_{n,d}^{\mathrm{ab}} \simeq (IO_n^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) \bigoplus \Gamma(n,d)^{\mathrm{ab}}.$$

Finally, in Section 5, we compute the integral homology groups of $IA_{2,p}$ for an odd prime p. In general, to compute the integral homology groups of $IA_{n,d}$ is quite difficult as well as that of IA_n. In the case where $n = 2$ and $d = p$, we can compute those as follows:

Theorem 1.3. *For any prime p,*

$$H_q(IA_{2,p}, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } q = 0, \\ \mathbf{Z}^{\oplus \alpha(p)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if } q = 1, \\ \mathbf{Z}^{\oplus(2\alpha(p)-2)} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3 \end{cases}$$

where $\alpha(p) = 1 + \frac{(p-1)p(p+1)}{12}$ is the rank of $\Gamma(2,p)$ as a free group.

We remark that $IO_{2,p}$ is isomrphic to the congruence subgroup $\Gamma(2,p)$ since $IA_2 = \mathrm{Inn}\, F_2$ due to Nielsen [12], and hence, it is a free group of rank $\alpha(p)$.

2. Preliminaries

In this section we review the IA-automorphism group of a free group and the first Johnson homomorphism of the automorphism group of a free group. Throughout this paper we use the following notation and conventions.

- The group $\mathrm{Aut}\, F_n$ acts on F_n from the right.
- For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ.
- For elements x and y of a group, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.1. **The IA-automorphism group.**

In this subsection, we prepare generators of IA_n, and some basic exact sequences which is required to prove our main theorems.

Let F_n be a free group on $\{x_1, \dots, x_n\}$. Magnus [11] showed that IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto {x_j}^{-1} x_i x_j, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct $i,\ j \in \{1, 2, \dots, n\}$ and

$$K_{klm} : \begin{cases} x_k & \mapsto x_k x_l x_m {x_l}^{-1} {x_m}^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq k) \end{cases}$$

for distinct $k,\ l,\ m \in \{1, 2, \dots, n\}$ such that $l < m$. Since IO_n is the qutient group $IA_n / \mathrm{Inn}\, F_n$, IO_n is also generated by (the coset classes of) automorphisms K_{ij} and K_{ijk}.

Next we give some basic exact sequences. Since the natural maps ρ and $\bar{\rho}$ are surjective, for any $n \geq 2$ and $d \geq 2$, we have exact sequences

$$1 \to IA_n \to IA_{n,d} \xrightarrow{\rho} \Gamma(n, d) \to 1 \tag{1}$$

and

$$1 \to IO_n \to IO_{n,d} \to \Gamma(n, d) \to 1 \tag{2}$$

respectively. Furthermore, by definition, we have

$$1 \to \mathrm{Inn}\, F_n \to IA_{n,d} \to IO_{n,d} \to 1. \tag{3}$$

These exact sequences are used in later sections.

2.2. **The first Johnson homomorphism.**

In this subsection, we review the first Johnson homomorphism of the automorphism group of a free group. For each $k \geq 1$, let $\Gamma_n(k)$ be the k-th subgroup of the lower central series of F_n defined by

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 1.$$

We denote the graded quotients $\Gamma_n(k)/\Gamma_n(k+1)$ by $\mathcal{L}_n(k)$. Set $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$. Then it is well known that $\mathcal{L}_n$ naturally has a structure of a graded Lie algebra over $\mathbf{Z}$ induced from the commtator bracket on F_n and it is naturally isomorphic to a graded free Lie algebra over H. In particular, we have $\mathcal{L}_n(1) = H$ and $\mathcal{L}_n(2) = \Lambda^2 H$. (For details, see [14].)

In this paper, for any $x \in \Gamma_n(k)$, we also denote by x the coset class of x in $\mathcal{L}_n(k)$.

For each $k \geq 1$, let

$$\tau' : IA_n \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(2)) = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

be the homomorphism defined by

$$\sigma \mapsto (x \mapsto x^{-1}x^{\sigma}).$$

for $\sigma \in IA_n$ and $x \in H$. The map τ' naturally induces a homomorphism

$$\tau : IA_n^{\mathrm{ab}} \to H^* \otimes_{\mathbf{Z}} \Lambda^2 H.$$

These homomorphisms τ' and τ are called the first Johnson homomorphisms of $\mathrm{Aut}\, F_n$. The map τ is a $GL(n, \mathbf{Z})$-equivariant isomorphism. In particular, IA_n^{ab} is a free abelian group of rank $\frac{1}{2}n^2(n-1)$. (See [8, Theorem 6.1].)

Next, we consider IO_n^{ab}. For any $y \in F_n$, we denote by ι_y the inner automorphism of F_n such that $x^{\iota_y} = y^{-1}xy$ for any $x \in F_n$. Considering a natural isomorphim $\mathrm{Inn}\, F_n \to F_n$; $\iota \mapsto y$, we often identify $\mathrm{Inn}\, F_n$ with F_n. Then we have $(\mathrm{Inn}\, F_n)^{\mathrm{ab}} = H$. It is also known that the induced homomorphism $(\mathrm{Inn}\, F_n)^{\mathrm{ab}} = H \to H^* \otimes_{\mathbf{Z}} \Lambda^2 H = IA_n^{\mathrm{ab}}$ from the inclusion map $\mathrm{Inn}\, F_n \hookrightarrow IA_n$ is injective, and whose image, which we identify with H by this map, is a direct summand of $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ as a $\mathbf{Z}$-module. Hence we see IO_n^{ab} is isomorphic to a free abelian group $(H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H$ of rank $\frac{1}{2}n(n+1)(n-2)$. For details, see [8, Theorem 6.2].

Now, the first Johnson homomorphism τ' induces a homomorphism

$$\tau'_d : IA_n \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

Finally, we recall that τ'_d is extended to a homomorphism from $IA_{n,d}$ to $(H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$. For any $\mathbf{Z}/d\mathbf{Z}$-valued Magnus expansion θ, Kawazumi [8] constructed a crossed homomorphism

$$\tau^{\theta} : \mathrm{Aut}\, F_n \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$$

and showed that if we restrict it to $IA_{n,d}$, then the map τ^{θ} is a homomorpism. Furthermore he also show that $\tau^{\theta} \equiv \tau'_d$ on IA_n. Especially the restriction $\tau^{\theta}|_{IA_n}$ is independent of the choice of the Magnus expansion θ. For details, see [8, Theorem 3.1].

3. The abelianization of $IA_{n,d}$.

In this section we give a proof of Theorem 1.1. First, we see that since the first Johnson homomorphism τ is a $GL(n,\mathbf{Z})$-equvariant isomorphism, τ induces a surjective homomorphism

$$\tilde{\tau} : H_0(\Gamma(n,d), IA_n^{\mathrm{ab}}) \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

To show $\tilde{\tau}$ is an isomorphism, we use

Lemma 3.1. *For $n \geq 2$ and $d \geq 2$, we have*

$$d\,[K_{ij}] = 0 \quad \text{and} \quad d\,[K_{klm}] = 0$$

in $H_0(\Gamma(n,d), IA_n^{\mathrm{ab}})$.

Then considering the homological five term exact sequence

$$\begin{aligned} H_2(IA_n, \mathbf{Z}) \to H_2(IA_{n,d}, \mathbf{Z}) \to H_0(\Gamma(n,d), IA_n^{\mathrm{ab}}) \\ \xrightarrow{\eta} IA_{n,d}^{\mathrm{ab}} \to \Gamma(n,d)^{\mathrm{ab}} \to 0. \end{aligned}$$

of (1), and a homomorphism

$$\tau^{\theta} : IA_{n,d} \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}$$

induced from a Magnus expansion θ, we see η is injective. Hence we have a split exact sequence

$$0 \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \xrightarrow{\eta} IA_n^{\mathrm{ab}} \to \Gamma(n,d)^{\mathrm{ab}} \to 0.$$

This completes the proof of Theorem 1.1.

Here we consider the case where d equals to an odd prime integer p. For $n \geq 3$, Lee and Szczarba [10] showed that the abelianization $\Gamma(n,p)^{\mathrm{ab}}$ of the congruence subgroup $\Gamma(n,p)$ is a $\mathbf{Z}/p\mathbf{Z}$-vector space of dimension n^2-1. For $n=2$, Frasch [5] showed that the congruence subgroup $\Gamma(2,p)$ is a free group of rank $\alpha(p) := 1 + \frac{(p-1)p(p+1)}{12}$. Furthermore Nielsen [12] showed that $IA_2 = \mathrm{Inn}\, F_2$. Hence we have

Corollary 3.1.

$$IA_{n,p}^{\mathrm{ab}} = \begin{cases} \mathbf{Z}^{\oplus \alpha(p)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if} \quad n = 2, \\ (\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n-1)(n^2+2n+2)} & \text{if} \quad n \geq 3. \end{cases}$$

4. The abelianization of $IO_{n,d}$.

In this section we give a proof of Theorem 1.2. Considering the homological five term exact sequence of (3), we have

$$H_2(IA_{n,d},\mathbf{Z}) \to H_2(IO_{n,d},\mathbf{Z}) \to H_0(IO_{n,d},(\mathrm{Inn}\,F_n)^{\mathrm{ab}}) \xrightarrow{\delta} IA_{n,d}^{\mathrm{ab}} \to IO_{n,d}^{\mathrm{ab}} \to 0.$$

Since H and $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ is free abelian groups, the injective homomorphism

$$H = (\mathrm{Inn}\,F_n)^{\mathrm{ab}} \hookrightarrow IA_n^{\mathrm{ab}} = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

induces an injective homomorphism

$$\psi_d : H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \hookrightarrow (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.$$

For each i, $1 \le i \le n$, set $\iota_i := \iota_{x_i}$. Then, $\mathrm{Inn}\,F_n$ is a free group on $\{\iota_1, \dots, \iota_n\}$.

Lemma 4.1. *For $n \ge 2$ and $d \ge 2$,*

$$d\,[\iota_i] = 0, \quad 1 \le i \le n$$

in $H_0(IO_{n,d},(\mathrm{Inn}\,F_n)^{\mathrm{ab}})$.

Hence there exists an isomorphism

$$\bar{\xi} : H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \to H_0(IO_{n,d},(\mathrm{Inn}\,F_n)^{\mathrm{ab}})$$

such that $\psi_d = \delta \circ \bar{\xi}$, and we have a short exact sequence

$$0 \to H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z} \xrightarrow{\delta} IA_{n,d}^{\mathrm{ab}} \to IO_{n,d}^{\mathrm{ab}} \to 0,$$

and hence

$$\begin{aligned}((H^* \otimes_{\mathbf{Z}} \Lambda^2 H)\otimes_{\mathbf{Z}}\mathbf{Z}/d\mathbf{Z}) \,/\, (H \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}) &\simeq ((H^* \otimes_{\mathbf{Z}} \Lambda^2 H)/H) \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}, \\ &\simeq IO_n^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/d\mathbf{Z}.\end{aligned}$$

This completes the proof of Theorem 1.2.

For $n \ge 2$ and an odd prime p, by an argument similar to that in Corollary 3.1, we obtain

Corollary 4.1.

$$IO_{n,p}^{\mathrm{ab}} = \begin{cases} \mathbf{Z}^{\oplus\,\alpha(p)} & \text{if} \quad n = 2, \\ (\mathbf{Z}/p\mathbf{Z})^{\oplus \frac{1}{2}(n+1)(n^2-2)} & \text{if} \quad n \ge 3. \end{cases}$$

5. The integral homology groups of $IA_{2,p}$

In this section, we compute the integral homology groups of $IA_{2,p}$ for any odd prime p. Since the groups IA_2 and $\Gamma(2,p)$ are free groups stated above, considering the homological Lyndon-Hochscild-Serre spectral sequence of (1) for $n = 2$ and $d = p$, we see the homological dimension of $IA_{2,p}$ is 2. On the other hand, since the first homology group $H_1(IA_{2,p}, \mathbf{Z})$ is obtained in Section 3, it suffices to compute the second homology group $H_2(IA_{2,p}, \mathbf{Z})$. Our result is

Theorem 5.1. *For any odd prime p,*

$$H_2(IA_{2,p}, \mathbf{Z}) = \mathbf{Z}^{\oplus(2\alpha(p)-2)}$$

where $\alpha(p) = 1 + \frac{(p-1)p(p+1)}{12}$.

To prove this theorem, first, we directly compute the second cohomology groups of $IA_{2,p}$. Then, using the universal coefficients theorem, we obtain the second homology group of $IA_{2,p}$.

Proposition 5.1. *For any odd prime integer p, we have*

$$H^2(IA_{2,p}, \mathbf{Z}) = \mathbf{Z}^{\oplus(2\alpha(p)-2)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2}.$$

Similarly, we obtain

Proposition 5.2. *For any odd prime integer p, we have*

$$H^2(IA_{2,p}, \mathbf{Z}/q\mathbf{Z}) \simeq \begin{cases} (\mathbf{Z}/q\mathbf{Z})^{\oplus(2\alpha(p)-2)} & \text{if } (q,p) = 1, \\ (\mathbf{Z}/q\mathbf{Z})^{\oplus(2\alpha(p)-2)} \oplus (\mathbf{Z}/p\mathbf{Z})^{\oplus 2} & \text{if } q = p^e. \end{cases}$$

Using Propositions 5.1 and 5.2, we obtain the second homology group $H_2(IA_{2,p}, \mathbf{Z})$ by the universal coefficients theorem.

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